C 3547

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Name.......................................

Reg. No....................................

# FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2016

(CUCSS)

Mathematics

## MT 4E 02—ALGEBRAIC NUMBER THEORY

Time : Three Hours

Maximum : 36 Weightage

*Standard notation as in prescribed text is followed.*

### Part A

*Answer* **all** *questions.*
*Each question carries weightage* 1.

1. Express $t_1^4 + t_2^4$ in terms of elementary symmetric polynomials $(n = 2)$.

2. Find the order of the group G/H where G is a free abelian group with basis $x, y, z$ and H is generated by $-2x, x + y, y + z$.

3. Find $\theta$ such that $Q(\theta) = Q\left(\sqrt{2}, \sqrt[3]{5}\right)$.

4. Show that an algebraic number is an algebraic integer if and only if (iff) its minimal polynomial over $Q$ has coefficients in $\mathbb{Z}$.

5. Let $K = Q(\zeta)$ where $\zeta = e^{2\pi i/5}$. Calculate $N_K(\alpha)$ and $T_K(\alpha)$ for $\alpha = \zeta^2$.

6. Let $x$ and $y$ be non-zero elements of a domain D. Prove that $x \mid y$ iff $\langle x \rangle \supseteq \langle y \rangle$.

7. Find a ring which is not noetherian.

8. Is $10 = (3+i)\times(3-i) = 2\times 5$ an example of non-unique factorization in $\mathbb{Z}[i]$ ? Give reasons for your answer.

9. True *or* False ?

   A fractional ideal of $\mathfrak{D}$ is a finitely generated $\mathfrak{D}$-submodule of K.

10. Prove : If $\sigma$ is a proper ideal of the ring of integers $\mathfrak{D}$ of the number field $K_1$ then $G^{-1}$ properly contains $\mathfrak{D}$ .

11. State Minkowskis theorem.

Turn over

12. Show that the quotient group is $\mathbb{R}/\mathbb{Z}$ is isomorphic to the circle group $S$.

13. Sketch the lattice $\mathbb{R}^2$ generated by (– 1, 2) and (2, 2) and a fundamental domain for the lattice.

14. Let $d$ be a squarefree positive integer and let $K = Q(\sqrt{d})$. Calculate $\sigma : K \rightarrow L^{st}$.

(14 × 1 = 14 weightage)

## Part B

*Answer any* **seven** *questions.*
*Each question carries weightage* 2.

15. Let G be a finitely generated abelian group with no non-zero elements of finite order. Prove that G must be a free group.

16. Prove that the set of algebraic numbers is a subfield of the complex field $\mathbb{C}$.

17. Let $K = Q(\theta)$ be a number field. Prove : If all k-conjugates of $\theta$ are real, then the discriminant of any basis is positive.

18. Let K be a number field of degree $n$-prove that the $\mathfrak{D}$, the ring of integers of K, is a free abelian group of rank $n$.

19. Let $d$ be a squarefree rational integer with $d \not\equiv 1 (\mathrm{mod}\, 4)$. Then prove that $\mathbb{Z}[\sqrt{d}]$ is the ring of integers of $Q(\sqrt{d})$.

20. Prove that the group of units of $Q(\sqrt{-3})$ is the group $\{\pm 1, \pm w, \pm w^2\}$ where $w = e^{2\pi i/3}$.

21. Prove that an integral domain $\mathfrak{D}$ is noetherian iff $\mathfrak{D}$ satisfies the maximal condition.

22. Prove that an ring of integers of $Q(\sqrt{-5})$ is not a unique factorization domain.

23. If $x, y, z$ are integers such that $x^2 + y^2 = z^2$, prove that atleast one of $x, y, z$ is a multiple of 3.

24. Prove : If $\alpha_1, \ldots, \alpha_n$ is a basis of the number field K over Q, then $\sigma(\alpha_1), \ldots \sigma(\alpha_n)$ are linearly independent over $\mathbb{R}$.

(7 × 2 = 14 weightage)

## Part C

*Answer any* **two** *questions.*
*Each question carries weightage 4.*

25. Let K be a number field. Then prove that there is an algebraic integer $\theta \in k$ such that $k = \mathbb{Q}(\theta)$.

26. Let $\zeta = e^{2\pi/p}$ where $p$ is an odd prime. Prove that $\mathbb{Z}[\zeta]$ is the ring of integers of $\mathbb{Q}[\zeta]$.

27. Let $\mathcal{D}$ be a domain in which factorization into irreducibles is possible. Prove that factorization into irreducibles is unique iff every irreducible is prime.

28. Prove that the equation $x^4 + y^4 = z^2$ has no integer solutions with $x, y, z \neq 0$.

(2 × 4 = 8 weightage)