C 60048 (Pages : 3) Name..............................

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SIXTH SEMESTER B.Sc. DEGREE EXAMINATION, MARCH 2019

(CUCBCSS)

Mathematics

MAT 6B 09—REAL ANALYSIS

Time : Three Hours Maximum : 120 Marks

Section A

Answer all the **twelve** *questions.*
Each question carries 1 mark.

1. Give an example of a real valued function with set of real numbers as domain which is nowhere continuous.
2. Define uniform continuous function.
3. State Maximum-Minimum Theorem.
4. Define a tagged partition of a closed and bounded interval in $\mathbb{R}$.
5. Let $f(x) = x^2$ for $x \in [0,4]$. Calculate the Riemann sum corresponding to the partition $P = (0, 1, 2, 4)$ with the tags at the right end points of the subintervals.
6. State Fundamental Theorem of Calculus (Second form).
7. Define uniform norm of a bounded function.
8. Give an example to show that pointwise convergence of sequence of functions need not imples uniform convergence.
9. State Cauchy Criterion for the uniform convergence of a series of functions.
10. Define improper integral of the second kind.
11. Show that $B(m, n) = B(n, m)$.
12. Define Gamma function.

($12 \times 1 = 12$ mark

Section B

Answer any **ten** *out of fourteen questions.*
Each question carries 4 marks.

13. Let I be a closed and bounded interval and let $f : I \to \mathbb{R}$ be continuous of I. If $k \in \mathbb{R}$ is any numb satisfying $\inf f(I) \le k \le \sup f(I)$, then show that there exists a number $c \in I$ such that $f(c)$ =

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14. Show that the equation $x = \cos x$ has a solution in the interval $[0, \pi]$.

15. If $f : A \to \mathbb{R}$ is uniformly continuous on a subset A of $\mathbb{R}$ and if (x_n) is a Cauchy sequence in A, then show that $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$.

16. If f and g are uniformly continuous on subset A of $\mathbb{R}$, show that $f + g$ is uniformly continuous on A.

17. Consider the function h defined by $h(x) = x + 1$ for $x \in [0, 1]$ rational and $h(x) = 0$ for $x \in [0, 1]$ irrational. Show that h is not Riemann integrable.

18. State and prove the Mean Value Theorem for Integrals.

19. State Lebesgue's Integrability Criterion.Using this discuss the Riemann integrability of any step function on $[a, b]$.

20. Let F and G be differentiable on $[a, b]$ and let $f = F'$ and $g = G'$ belongs to $R[a, b]$, then show that $\int_a^b f\,G = [FG]_a^b - \int_a^b F\,g$.

21. Show that $\lim(x/(x+n)) = 0$ for all $x \in \mathbb{R}, x \geq 0$. Also show that the convergence is not uniform on the interval $[0, \infty)$.

22. Discuss the convergence and uniform convergence of $\sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}$.

23. Show that $\int_1^{\infty} \frac{1}{x^{4/3}}\,dx$ converges.

24. Show that $\int_0^1 \frac{\sin x}{\sqrt{x}}\,dx$ converges.

25. Show that $\Gamma(1/2) = \sqrt{\pi}$.

(10 × 4 = 40 marks

Section C

Answer any **six** *out of nine questions.*
Each question carries 7 marks.

State and prove Boundedness Theorem.

If $f(x) = x$ and $g(x) = \sin x$, show that both f and g are uniformly continuous of $\mathbb{R}$, but that thei product fg is not uniformly continuous on $\mathbb{R}$.

If $f \to \mathbb{R}$ is monotone on $[a, b]$, then show that $f \in \mathbb{R}[a, b]$.

29. State and prove Fundamental Theorem of Calculus (First Form).

30. Let (f_n) be a sequence of bounded functions on $A \subseteq \mathbb{R}$. Then show that this sequence converges uniformly on A to a bounded function f if and only if for each $\varepsilon > 0$ there is a number $H(\varepsilon)$ in N such that for all $m, n \geq H(\varepsilon)$ then $\| f_m - f_n \| \leq \varepsilon$.

31. Show that the uniform limit of a sequence of continuous real valued functions on a subset of real numbers is continuous.

32. Show that the improper integral $\int_0^1 \frac{1}{x} dx$ diverges.

33. Let f be a non-increasing function on $[1, \infty)$ such that $f(x) \geq 0 \ (l \leq x < \infty)$. Then show that $\sum_1^\infty f(n)$ will diverge if $\int_1^\infty f(x)\, dx$ diverges.

34. Derive a relation between Beta and Gamma function.

(6 × 7 = 42 marks)

Section D

Answer any **two** *out of three questions.*
Each question carries 13 *marks.*

35. Show that a continuous function on a closed and bounded interval I can be approximated arbitrarily closely by step function.

36. Let $f : [a, b] \to \mathbb{R}$ and let $c \in (a, b)$. Then show that $f \in \mathcal{R}[a, b]$ if and only if its restriction to $[a, c]$ and $[c, b]$ are both Riemann integrable. In this case show that $\int_a^b f = \int_a^c f + \int_c^b f$.

37. (a) Let $f_n(x) := 1/(1 + x)^n$ for $x \in [0, l]$. Find the pointwise limit f of the sequence f_n on [0, 1]. Does f_n converge uniformly to f on [0, 1] ?

(b) Let (c_n) be a decreasing sequence of positive numbers. If $\sum c_n \sin nx$ is uniformly convergent, then show that $\lim (nc_n) = 0$.

(2 × 13 = 26 marks)